A Generalized Effective Interaction

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We discuss the possibility of neglecting poles of the effective interaction defined in a previous paper. In order to use model spaces which are not completely known we generalized the definition of the effective interaction considering as an example the self energy operator for the retarded Green-function.

1. Introduction

This is a second paper in which we investigate the characteristics of effective interactions in nuclear models, used in calculating matrix elements of the Green-operator. In the first paper [1] we discussed the connection between the size and the position of the energy interval in which the model should be valid and the space of the model states. Two problems are to be discussed in this paper.

First we examine the possibility of neglecting poles of the effective interaction within the working interval and the consequences of such a procedure.

Secondly, the effective interaction [2–4] defined in A requires exact knowledge of the model states; for example to form the matrix elements of the unperturbed Green-operator $G^0(w)$. We generalize our method, to a model space with unknown model states in Section 3.

In Section 4 we use, as an example, the model space spanned with the states $a_i^+ \,|\, 0\rangle$. Here, $|\,0\rangle$ is the exact ground state of the system with $N-1$ nucleons. We attain information concerning the energy dependence of the "self energy operator" of the retarded Green-function, which can be understood as the one particle potential in the shell model for nuclei.

In another paper [5] we already used the quasiparticle-quasihole space in order to investigate the energy dependence of the effective interaction in MIGDAL's [6] quasiparticle theory.

2. Definition of the Effective Interaction; Neglection of Poles

In A we defined the effective interaction T^1

$$T^1(W) = V + V\,[W - H_0 - c\,V^1]^{-1}\,T^1(W) = V + V[W - H - c\,V^1]^{-1}\,V. \tag{2.1}$$

Here we have split the Hamiltonian H in an unperturbed part H_0 and the residual interaction V. The operator V^1 has to be chosen in such a way that the effective interaction is weakly energy dependent in the interesting domain

$T^1(W)$ is used in calculating the T operator

$$T(W) = V + V[W - H_0]^{-1}\,T(W);$$
$$T(W) = T^1(W) + T^1(W)\left[\frac{1}{W - H_0} - \frac{1}{W - H_0 - c\,V^1}\right] T(W). \tag{2.2}$$

The poles W_n of $T^1(W)$ appear as the eigenvalue of the operator H^1

$$H^1 = H + c\,V^1;$$
$$H^1\,|\,E_n{}^1\rangle = E_n{}^1\,|\,E_n{}^1\rangle; \quad W_n = E_n{}^1. \tag{2.3}$$

We select the unperturbed Hamiltonian H_0, in such a way, that it commutes with the operator V^1

$$[H_0, V^1] = 0. \tag{2.4}$$

The equations then become greatly simplified, if we allow the parameter c to move towards infinity

$$T^1(W) = V + V\,\frac{1-P}{W - H_0}\,T^1(W),$$
$$T(W) = T^1(W) + T^1\,\frac{P}{W - H_0}\,T(W), \tag{2.5}$$
$$V^1\,|\,i\rangle = V_i{}^1\,|\,i\rangle; \quad i = 1, 2, \ldots, N; \quad P = \sum_{i=1}^{N} |\,i\rangle\,\langle i\,|.$$

The eigenvalues of the operator V^1 span the model space which is generally finite. The projection operator P, is the unity operator in model space.

In order to clarify the physical meaning of an infinite interaction $c\,V^1$, we mention the possibility of splitting H^1 into a finite and an infinite part

$$H^1 = H_f{}^1 + H_i{}^1. \tag{2.6}$$

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Assuming for the moment that the Hamiltonian H is finite, H_i^1 has model states as eigenstates

$$[H_i^1, V^1] = [H_i^1, P] = 0 \tag{2.7}$$

but of course with infinite eigenvalues. The finite part H_f^1 has the form

$$H_f^1 = Q\,H\,Q; \qquad Q = 1 - P \tag{2.8}$$

which means that the finite eigenstates of H^1 span the space which is orthogonal to the model space. Using (2.1) together with (2.7) and (2.8) we get for $T^1(W)$

$$\begin{aligned} T^1(W) &= V + V\,\frac{1}{W - Q\,H\,Q - H_i^1}\,V \\ &= V + V\,Q\,\frac{1}{W - Q\,H\,Q}\,Q\,V \end{aligned} \tag{2.10}$$

which has now the same form as the effective interaction in the reaction theory of FESHBACH [7] and is equivalent to the representation given by MÜLLER [8]. Now, we may also compute matrix elements of the Green-operator $G(W)$ in the model space.

$$P\,G(W)\,P = P\,G_0(W) + G_0(W)\,P\,T^1(W)\,P\,G(W)\,P.$$

For the matrix elements this means

$$\begin{aligned} \langle i\,|\,G(W)\,|\,i'\rangle &= \langle i\,|\,G_0(W)\,|\,i'\rangle \\ &+ \sum_{i''=1}^{N} \langle i\,|\,G_0(W)\,|\,i\rangle\,\langle i\,|\,T^1(W)\,|\,i''\rangle\,\langle i''\,|\,G(W)\,|\,i'\rangle. \end{aligned} \tag{2.11}$$

In order to make practical use of T^1, in calculating matrix elements of the Green-operator or the T operator in the energy interval I

$$W_1 < W < W_2, \tag{2.12}$$

we must demand a weak energy dependence of $T^1(W)$ in this interval. If we thus replace $T^1(W)$ in the interval I by an energy independent effective interaction, it is sufficient for the calculation of the approximate energy eigenvalues. In determining the residues of the matrix elements of the Green-operator, we need the linear term of the Taylor-series for T^1

$$T^1(W) = T_0^1 + (W - W_0)\,T_1^1, \tag{2.13}$$

in which W_0 is one point in I. Consequently, the operator $c\,V^1$ must be chosen in such a way, that no poles of $T^1(W)$ appear within or nearly the interval I. The problem of the choice of V^1 has been discussed in detail in A.

Practically only specific kinds of states can be described within a certain model. Let us now assume that we have an eigenvalue E_{n_0} of H within the interval I, while the eigenstate $|\,E_{n_0}\rangle$ cannot be approximated by a linear combination of model states. Consequently, a corresponding eigenvalue $E_{n_0}{}^1$ of H^1 is situated in or near I. One can easily see that the states $|\,E_{n_0}\rangle$ and $|\,E_{n_0}{}^1\rangle$ are similar

$$\langle E_{n_0}\,|\,E_{n_0}{}^1\rangle \sim 1\,. \tag{2.14}$$

Following (2.5) the effective interaction has a pole within or near the interval I. In order to eliminate this difficulty, we examine the system with the reduced Hamiltonian H' instead of the original system. H' should have the same eigenstates and eigenvalues of H, varying in that one eigenvalue E_n' is shifted toward infinity.

$$H' = H + \lim_{c\to\infty} c\,|\,E_{n_0}\rangle\,\langle E_{n_0}\,|\,. \tag{2.15}$$

The effective interaction, which we can now construct for the reduced system using our model space, has no poles within or near I. Two possibilities follow:

a) E_{n_0} lies outside of I. Then a matrix element of the Green-operators $G(W)$ and $G'(W)$ differs by a function $f(W)$

$$\begin{aligned} f(W) = \langle i\,|\,[W - H]^{-1} \\ - [W - H - \lim_{c\to\infty} c\,|\,E_{n_0}\rangle\,\langle E_{n_0}\,|\,]^{-1}\,|\,i'\rangle \end{aligned} \tag{2.16}$$

which has no poles in I; the poles and the residues of the two Green-operators are equal in model space.

b) E_{n_0} lies within the interval. In this case, $f(W)$ has a pole at the point $W = E_{n_0}$. This pole is absent in the matrix elements of $G'(W)$, and is not described by our reduced model. All other eigenvalues in I are reproduced correctly by the reduced model.

One may naturally extend this method, and proceed to shift an even larger number of eigenvalues towards infinity in the reduced system. Strictly speaking, this is possible for all eigenvalues which cannot be described within the model.

3. The Diagonalization of the Green Operator in Model Space; Other Generalized Effective Interactions

We now extend our theory somewhat. Following equation (2.4), we have always taken for granted that the model states $|\,\lambda\rangle$ in the operator V^1

$$V^1 = \sum_{\lambda} |\,\lambda\rangle\,\langle\lambda\,| \tag{3.1}$$

are also eigenstates of H_0. Let us now assume that the eigenstates of H_0

$$H_0 \,|\, E_\lambda^0\rangle = E_\lambda^0 \,|\, E_\lambda^0\rangle \tag{3.2}$$

and the model states $|\,\lambda\rangle$ can be characterized by the same quantum numbers λ, although they are not the same states. We show that the poles W_n of the effective interaction m, defined by

$$\begin{aligned}\langle\lambda\,|\,G(W)\,|\,\lambda'\rangle &= \langle E_\lambda^0\,|\,G^0(W)\,|\,E_{\lambda'}{}^0\rangle \\ &+ \sum_{\lambda''}\langle E_\lambda^0\,|\,G^0(W)\,|\,E_\lambda^0\rangle m_{\lambda\lambda''}(W)\,\langle\lambda''\,|\,G(W)\,|\,\lambda'\rangle\,,\end{aligned} \tag{3.3}$$

occur at the eigenvalues of

$$H^1 = \lim_{c\to\infty}(H + c\,V^1)\,; \tag{3.4}$$

$$H^1\,|\,E_n{}^1\rangle = E_n{}^1\,|\,E_n{}^1\rangle, \qquad W_n = E_n{}^1. \tag{3.5}$$

The Eq. (3.5) is analogous to condition (2.3), but we should keep in mind that the model states $|\,\lambda\rangle$ are not necessarily orthogonal, although V^1 is always positive definit. To show condition (3.5) we examine the eigenvalue equation

$$\frac{1}{W-H}\,V^1\,|\,\eta_\lambda(W)\rangle = \eta_\lambda(W)\,|\,\eta_\lambda(W)\rangle\,. \tag{3.6}$$

The roots of $\eta_\lambda(W)$ appear at the eigenvalues of H^1. The eigenstates $|\,\eta_\lambda(W)\rangle$ should fulfill the usual normalization relation

$$\begin{aligned}&\langle\eta_\lambda(W)\,|\,V^1\,|\,\eta_{\lambda'}(W)\rangle \\ &\quad= \sum_{\lambda''}\langle\eta_\lambda(W)\,|\,\lambda''\rangle\,\langle\lambda''\,|\,\eta_{\lambda'}(W)\rangle = \delta_{\lambda\lambda'}\,.\end{aligned} \tag{3.7}$$

Using the unitary transformation with the matrix U

$$U_{\lambda\lambda'}(W) = \langle\lambda\,|\,\eta_{\lambda'}(W)\rangle\,, \tag{3.8}$$

we can diagonalize the Green-operator within the model space.

$$\begin{aligned}&\sum_{\lambda'''\lambda'}\langle\eta_\lambda(W)\,|\,\lambda'''\rangle\langle\lambda'''\,|\,G(W)\,|\,\lambda''\rangle\;\langle\lambda''\,|\,\eta_{\lambda'}(W)\rangle \\ &\qquad= \eta_\lambda(W)\;\delta_{\lambda\lambda'}\,.\end{aligned} \tag{3.9}$$

We see that $\eta_\lambda(W)$ is the eigenvalue of the Green-operator in the model space. Naming the Green-operator $g(\mathrm{W})$ in model space

$$g_{\lambda\lambda'}(W) = \langle\lambda\,|\,G(W)\,|\,\lambda'\rangle \tag{3.10}$$

and with h_0 and $g_0(W)$ the unperturbed Hamiltonian and Green-operator in the corresponding space of the states $|\,E_\lambda^0\rangle$ we can present (3.3) in the following way:

$$g_{\lambda\lambda'}(W) = g_{0\lambda\lambda'}(W) + \sum_{\lambda''} g_{0\lambda\lambda}(W)\,m_{\lambda\lambda''}(W)\,g_{\lambda''\lambda'}(W)\,. \tag{3.11}$$

The solution of (3.11)

$$\begin{aligned}g(W) &= [g_0{}^{-1}(W) - m(W)]^{-1} \\ &= [W - h_0 - m(W)]^{-1}\end{aligned} \tag{3.12}$$

shows us, that the poles of $m(W)$ occur at the roots of the eigenvalues $\eta_\lambda(W)$. However, these roots are also the eigenvalues of H^1.

A weak energy dependence of the effective interaction m in the interval I, means that the states $|\,\eta_\lambda(W)\rangle$ are also weakly energy dependent within I. In this case, and for each eigenstate $|\,E_\lambda\rangle$ and eigenvalue E_λ of the Hamiltonian H within I, a corresponding eigenstate $|\,\eta_\lambda(W)\rangle$ with the property

$$|\,\eta_\lambda(E_\lambda)\rangle = |\,E_\lambda\rangle \approx |\,\eta_\lambda(W)\rangle \quad \text{for } W \text{ in the interval I,} \tag{3.13}$$

exists. Thus, the states

$$|\,\lambda(W)\rangle = \sum_{\lambda'}\langle\lambda'\,|\,\eta_\lambda(W)\rangle\,|\,\lambda'\rangle = \sum_{\lambda'} U_{\lambda\lambda'}(W)\,|\,\lambda'\rangle \tag{3.14}$$

are approximately the projections of the states $|\,E_\lambda\rangle$ into the model space. Our method differs from the usual variational method (Tamm-Dancoff equations etc.) as we will show now. To simplify the discussion, let us assume normalized model states

$$\langle\lambda\,|\,\lambda'\rangle = \delta_{\lambda\lambda'}; \qquad V^1 = P \tag{3.15}$$

(P is the projection operator into the model space). The states $|\,\lambda\rangle$ are now also normalized

$$\langle\lambda\,|\,\lambda'\rangle = \delta_{\lambda\lambda'} \tag{3.16}$$

as we can see from (3.7). If we solve the variational problem

$$\frac{\langle E^\nu\,|\,H\,|\,E^\nu\rangle}{\langle E^\nu\,|\,E^\nu\rangle} = \text{minimum}; \quad P\,|\,E^\nu\rangle = |\,E^\nu\rangle\,, \tag{3.17}$$

we have to diagonalize the Hamiltoniean in the model space

$$P\,H\,P\,|\,E_{\mathrm{i}}{}^\nu\rangle = E_{\mathrm{i}}{}^\nu\,|\,E_{\mathrm{i}}{}^\nu\rangle \tag{3.18}$$

The energy eigenvalues $E_{\mathrm{i}}{}^\nu$ are upper bonds for the exact energies

$$E_{\mathrm{i}} < E_{\mathrm{i}}{}^\nu\,. \tag{3.19}$$

The states $|\,\lambda(W)\rangle$ on the other hand, are the eigenstates of the operator

$$P\,\frac{1}{W-H}\,P = \frac{1}{W-h_0-m(W)} \tag{3.20}$$

as we can see by multiplying Eq. (3.6) from the left by P.

The choice of the unperturbed Hamiltonian h_0 in the model space influences the form and therefore

the relative energy dependence of the effective interaction $m(W)$. But as we can see from Eq. (3.20) there is no influence on the absolute W-dependence of $m(W)$, which is only important for the validity of the model.

4. The Energy Dependence of the "Self-Energy Operator" for the Retarded Green Function

Let us use the states

$$|\lambda\rangle = a_\lambda^+ |0, N-1\rangle; \qquad a_\lambda^+ = \int \mathrm{d}^3x\, \varphi_\lambda^*(\boldsymbol{x})\, \psi^+(\boldsymbol{x}) \tag{4.1}$$

as model states [$\psi^+(\boldsymbol{x})$ creates a particle at the point $\boldsymbol{x}$; $|0, \mathrm{N}-1\rangle$ is the exact ground state of the $N-1$ particle system].

A specific choice of the wave functions φ_λ has no influence on the model space. We choose them in such a way, that H_0 becomes diagonal

$$\begin{aligned} \varphi_{\boldsymbol{k}}(x) = \frac{1}{V\Omega} \exp\{i\,\boldsymbol{k}\,\boldsymbol{x}\}; \quad H_0 = E_0^0 + \sum_{\boldsymbol{k}} a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}} \\ H_0 |E_0^0\rangle = E_0^0 |E_0^0\rangle; H_0 a_{\boldsymbol{k}}^+ |E_0^0\rangle \\ = \left(E_0^0 + \frac{k^2}{2M}\right) a_{\boldsymbol{k}}^+ |E_0^0. \end{aligned} \tag{4.2}$$

Here, $|E_0^0\rangle$ is the ground state of the unperturbed system. Following (3.3) we can introduce the "self-energy operator" m of the retarded Green-function

$$\begin{aligned} \langle 0 | a_{k_1} G(W)\, a_{k_2}^+ | 0\rangle = \langle E_0^0 | a_{k_1}\, G^0(W)\, a_{k_2}^+ | E_0^0\rangle \\ + \sum_{k_1'} \langle E_0^0 | a_{k_1} G^0(W)\, a_{k_1}^+ | E_0^0\rangle\, m_{k_1 k_1'} \\ \cdot (W) \langle 0 | a_k G(W)\, a_{k_2}^+ | 0\rangle . \end{aligned} \tag{4.3}$$

Symbolically, we can write

$$g(W) = g^0(W) + g^0(W)\, m(W)\, g(W); \tag{4.4}$$

The poles of the "self-energy operator" $m(W)$, appear at the eigenvalues of H^1

$$H^1 = H + \lim_{c\to\infty} c\, V^1; \quad V^1 = \sum_{\boldsymbol{k}} a_{\boldsymbol{k}_1}^+ |0\rangle \langle 0 | a_{\boldsymbol{k}_1}.$$

The operator V^1 is positiv definit. The energy eigenvalues E_λ are given by the value of W for which the operator

$$W - h_0 - m(W) \tag{4.5}$$

has the eigenvalue zero. Using an expansion of $m(W) - E_0$, on the groundstate energy E_0

$$m(W) - E_0 = \mathrm{m}^0 + (W - E_0)\, m^1, \tag{4.6}$$

the excitation energies

$$\varepsilon_\lambda = E_\lambda - E_0 \tag{4.7}$$

of the system are given as eigenvalues of a one particle Hamilton-operator

$$\left(\frac{p^2}{2m^*} - U\right) \varphi_\lambda(x) = \varepsilon_\lambda\, \varphi_\lambda(x). \tag{4.8}$$

The effective mass m^*, and the one particle potential U, are defined by the equations

$$m/m^* = 1 - m^1; \qquad U = m^0(1 - m^1). \tag{4.9}$$

Usually, the operator m/m^* is approximated by the constant a. We can see, examining residues of the Green-functions, that a is the so-called quasiparticle residue. In our approximation for m^*/m we have

$$\begin{aligned} \langle 0 | a_\lambda \frac{1}{W-H} a_\lambda^+ | 0\rangle = \frac{a}{W - E_0 - \varepsilon_\lambda}; \\ a_\lambda = \int \mathrm{d}^3x\, \varphi_\lambda(x)\, \psi(x). \end{aligned}$$

for $$W \to E_0 + \varepsilon_\lambda$$

Similar to the way, the finite Fermi momentum k_F limits the quasiparticle excitations to momenta outside of the Fermi sphere in the infinite system, the permitted excitation energies in the finite system become limited by the condition

$$\varepsilon_\lambda \geqq E_0(N) - E_0(N-1). \tag{4.10}$$

6. Summary and Discussion

In this paper we were primarily concerned with the following problems.

If $T^1(w)$, the effective interaction, has poles within or below the considered energy interval, we are able to help ourselves by using a reduced model in which we exclude the calculation of the corresponding eigenvalues E_λ, of the Hamiltonian H. Generally, we can say that it is possible to compute eigenvalues and eigenstates in a model, if the projection of these states into the model space is large. Eigenenergies which lie within the working interval, and which have a small projection into the model space, either yield a strong energy dependence of the effective interaction, or these eigenvalues are omitted in a reduced model, and therefore cannot be calculated.

If we do not calculate the effective interaction in using perturbation theory, but would rather fit m on experimental dates, we have extended our method for such "unknown model spaces". Principally, we may now choose the model space S at random. However, S determines the form of m, and the matrix

elements of the Green-operator which we can calculate with our model. If, for example, m should be an one particle operator, the characterizing quantum numbers of the model states must be the same as for a one particle state.

In Sect. 4 we considered an example for such a model space. In this case, the self energy operator of the retarded Green-function is the effective interaction. Migdal used the self energy operator to give a new justification of the one particle model. Here, we do not want to prefer his justification, but would rather state that there are various possibilities.

In another paper [5] we used the quasiparticle quasihole space as modelspace to establish a model to determine excitation energies and transition probabilities of atomic nuclei. The equations derived show a remarkable correspondence to those in MIGDAL's [6] quasiparticle theory.

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Induzierte Stoßprozesse

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Stimulated Collision Processes

Similar to the stimulated emission in radiation processes there are stimulated collisions in collision processes too. These stimulated collisions do not lead to an amplification of the electron transitions like in radiation processes but attenuate the transitions. So the occurance of stimulated collisions can stabilize the population density of the levels.

1. Einleitung

Nachdem bei den Strahlungsprozessen durch den Bau von Lasern und Masern die induzierte Emission eine große Bedeutung erlangt hat, ist es angebracht zu untersuchen, ob auch bei Stoßprozessen induzierte Übergänge auftreten können und was sie bewirken können.

Historisch gesehen war die Situation, die zu der Auffindung der induzierten Emission führte, anders als sie jetzt bei den Stoßprozessen vorliegt. EINSTEIN versuchte mit seinen Veröffentlichungen [1, 2] eine neue Ableitung der kurz vorher gefundenen Strahlungsformel zu geben, und zwar durch Einführung von Hypothesen über die Ein- und Ausstrahlung der Atome oder Moleküle. Bei den Elektron-Atomstößen dagegen kann man heute davon ausgehen, daß die Energieverteilungsfunktion der Elektronen im thermodynamischen Gleichgewicht (Maxwell-Verteilung bei Voraussetzung der klassischen Statistik und Fermi-Verteilung bei Voraussetzung der Quantenstatistik) als allgemein bekannt und gesichert angesehen werden können. Das ermöglicht bei den Stößen ein anderes Vorgehen als das von Einstein. Durch Vorgabe der Fermi-Verteilung sowie der bekannten Ausdrücke für die Elektronenstoßanregung und der Stöße zweiter Art läßt sich die Existenz von induzierten Stößen und eine Beziehung zur Berechnung ihrer Größe herleiten (Abschn. 3). Zum Beweis, daß dieser Weg gangbar ist, ist es zweckmäßig Analogien zwischen Stoß- und Strahlungsprozessen herauszuarbeiten und zu zeigen, daß die induzierte Emission auf eine ähnliche Weise (unter Vorgabe der Planckschen Formel und der Ausdrücke von spontaner Emission und normaler Absorption) herleitbar ist (Abschn. 2). Bei dieser Gegenüberstellung entspricht der Fermi-Verteilung die Plancksche Formel und da für $e^{E/kT} \gg 1$ die Fermi-Verteilung in die Maxwell-Verteilung und die Plancksche For-

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